Partial Solution Set, Leon §7.5

7.5.3a Given the vector $\mathbf{x} = (8, -1, -4)^T$, find a Householder transformation H such that $H\mathbf{x} = \|\mathbf{x}\|\mathbf{e}_1$.

Solution: Following the labels used in the text, we have $\alpha = \|\mathbf{x}\|_2 = 9$, $\beta = \alpha(\alpha - x_1) = 9$, and $\mathbf{v} = (-1, -1, -4)^T$. The matrix H is given by

$$H = I - \frac{1}{\beta}\mathbf{v}\mathbf{v}^T = \frac{1}{9}(9I - \mathbf{v}\mathbf{v}^T) = \frac{1}{9}\begin{bmatrix} 8 & -1 & -4 \\ -1 & 8 & -4 \\ -4 & -4 & -7 \end{bmatrix}.$$

It is easy to verify that $H\mathbf{x} = (9, 0, 0)^T$.

7.5.4b Find a Householder transformation that zeros out the last two coordinates of the vector $\mathbf{x} = (4, -3, -2, -1, 2)^T$.

Solution: Since we will be preserving the first two coordinates of $\mathbf{x}$, it follows that the matrix H in question is of the form

$$H = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & H' \end{bmatrix},$$

where H' is a 3×3 Householder matrix constructed to zero all but the first entry of $\mathbf{x}' = (-2, -1, 2)^T$.

Using the author's notation, we have the following:

$\alpha = \|\mathbf{x}\|_2 = 3$

$\beta = \alpha(\alpha - x_1) = 15$

$\mathbf{v} = \mathbf{x} - (\alpha, 0, 0)^T = (-5, -1, 2)$

$H' = I - \frac{1}{\beta}\mathbf{v}\mathbf{v}^T = \frac{1}{15}(15I - \mathbf{v}\mathbf{v}^T) = \frac{1}{15}\begin{bmatrix} -10 & -5 & 10 \\ -5 & 14 & 2 \\ 10 & 2 & 11 \end{bmatrix}.$

It follows that $H = \dfrac{1}{15}\begin{bmatrix} 15 & 0 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 & 0 \\ 0 & 0 & -10 & -5 & 10 \\ 0 & 0 & -5 & 14 & 2 \\ 0 & 0 & 10 & 2 & 11 \end{bmatrix}.$

7.5.5 Given $A = \begin{bmatrix} 3 & 3 & -2 \\ 1 & 1 & 1 \\ 1 & -5 & 1 \\ 5 & -1 & 2 \end{bmatrix}$,

1. Determine the scalar β and the vector $\mathbf{v}$ for the Householder matrix $H = I - (1/\beta)\mathbf{v}\mathbf{v}^T$ that zeros out the last three entries of A.

   Solution: We start with $\alpha = \|\mathbf{a}_1\| = 6$. Then we find $\beta = \alpha(\alpha - 3) = 18$. Finally, $\mathbf{v} = \mathbf{a}_1 - \alpha\mathbf{e}_1 = (-3, 1, 1, 5)^T$.

2. Without explicitly forming the matrix H, compute HA.

   The first column of HA is $H\mathbf{a}_1$, which (by construction of H) is $(6, 0, 0, 0)^T$. The second column of HA is $H\mathbf{a}_2 = (I - (1/18)(\mathbf{v}^T\mathbf{a}_2)\mathbf{v} = \mathbf{a}_2 + \mathbf{v} = (0, 2, -4, 4)^T$. The last column of HA is $H\mathbf{a}_3 = (I - (1/18)(\mathbf{v}^T\mathbf{a}_3)\mathbf{v} = \mathbf{a}_3 - \mathbf{v} = (1, 0, 0, -3)^T)$. So

$$HA = \begin{bmatrix} 6 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & 4 & -3 \end{bmatrix}.$$

7.5.6 Let $A = \begin{bmatrix} 1 & 2 & -4 \\ 2 & 6 & 7 \\ -2 & 1 & 8 \end{bmatrix}$, and $\mathbf{b} = (9, 9, -3)^T$.

1. Use Householder transformations H_1 and H_2 to transform A into an upper triangular matrix R. Also transform $\mathbf{b}$, i.e., compute $\mathbf{b}^{(1)} = H_2 H_1 \mathbf{b}$.
2. Solve $R\mathbf{x} = \mathbf{b}^{(1)}$ for $\mathbf{x}$, and check your answer by computing the residual $\mathbf{b} - A\mathbf{x}$.

Solution:

1. We begin by constructing a transformation H_1 to transform $\mathbf{x_1} = (1, 2, -2)^T$ to a vector of the form $(\alpha, 0, 0)^T$. We have $\alpha = \|\mathbf{x}_1\| = 3$, $\beta = 3(3 - 1) = 6$, $\mathbf{v} = (x_1 - \alpha, x_2, x_3)^T = (-2, 2, -2)^T$, and $H_1 = I - \frac{1}{\beta}\mathbf{v}\mathbf{v}^T = \frac{1}{6}\begin{bmatrix} 2 & 4 & -4 \\ 4 & 2 & 4 \\ -4 & 4 & 2 \end{bmatrix}$. We use H_1 to introduce zeros below the first pivot entry in A:

$$H_1 A = \frac{1}{6}\begin{bmatrix} 2 & 4 & -4 \\ 4 & 2 & 4 \\ -4 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -4 \\ 2 & 6 & 7 \\ -2 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -2 \\ 0 & 4 & 5 \\ 0 & 3 & 10 \end{bmatrix}.$$

   Our next goal is to introduce 0's below the main diagonal in column two. To accomplish this, we first construct a 2×2 Householder transformation H to transform $\mathbf{x}_2 = (4, 3)^T$ to a vector of the form $(\alpha, 0)^T$. We have $\alpha = \|\mathbf{x}_2\| = 5$, $\beta = 5(5 - 4) = 5$, $\mathbf{v} = (-1, 3)^T$, and $H = I - \frac{1}{5}\mathbf{v}\mathbf{v}^T = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$. We now construct $H_2 = \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline \mathbf{0} & H \end{array}\right] = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 3 & -4 \end{bmatrix}$. Finally, we compute the products

$$R = H_2\,(H_1 A) = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 3 & -4 \end{bmatrix}\begin{bmatrix} 3 & 4 & -2 \\ 0 & 4 & 5 \\ 0 & 3 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -2 \\ 0 & 5 & 10 \\ 0 & 0 & -5 \end{bmatrix}$$

and

$$\mathbf{b}^{(1)} = H_2 H_1 \mathbf{b} = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 3 & -4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 2 & 4 & -4 \\ 4 & 2 & 4 \\ -4 & 4 & 2 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ -3 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 5 \end{bmatrix}.$$

2. Solving the triangular system $R\mathbf{x} = \mathbf{b}^{(1)}$, we get $\mathbf{x} = (-1, 3, -1)^T$. The residual is $\mathbf{r} = \mathbf{b} - A\mathbf{x} = \mathbf{0}$.

7.5.13 Let $\mathbf{u}$ be a unit vector in $\mathbf{C}^n$, and let $U = I - 2\mathbf{u}\mathbf{u}^H$.

1. Show that $\mathbf{u}$ is an eigenvector of U. What is the corresponding eigenvalue?
2. Let $\mathbf{z}$ be a nonzero vector in $\mathbf{C}^n$ that is orthogonal to $\mathbf{u}$. Show that $\mathbf{z}$ is also an eigenvector of U. What is the corresponding eigenvalue?

Solution:

1. It suffices to compute

$$U\mathbf{u} = \left(I - 2\mathbf{u}\mathbf{u}^H\right)\mathbf{u} = \mathbf{u} - 2\mathbf{u}\mathbf{u}^H\mathbf{u} = -\mathbf{u},$$

from which we can see that $\mathbf{u}$ is an eigenvector with associated eigenvalue $\lambda = -1$.

2. As in part (a), we compute

$$U\mathbf{z} = \left(I - 2\mathbf{u}\mathbf{u}^H\right)\mathbf{z} = \mathbf{z} - 2\mathbf{u}\mathbf{u}^H\mathbf{z} = \mathbf{z},$$

showing that $\mathbf{z}$ is an eigenvector with associated eigenvalue $\lambda = 1$.